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Mandelstam Representation for the Bethe–Salpeter Amplitudes†

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Abstract

Mandelstam representation is shown to be valid for the total Bethe-Salpeter scattering amplitude in the ladder approximation.

1. *Introduction*

In potential scattering, our knowledge in the analyticity properties of the scattering amplitudes is in a very satisfactory condition. We know that for a large class of potentials, the scattering amplitude satisfies Khuri's (1957) fixed-t dispersion relation and has a nonshrinking Martin domain (Martin, 1966) in the momentum transfer t-plane. Starting from these properties, one can also arrive (Bowcock & Martin, 1959; Blankenbecler *et al.*, 1960; Cheung, 1969) at the Mandelstam representation (Mandelstam, 1958) quite readily if the elastic unitarity condition is fully utilized. On the other hand, we are in quite a different situation with the Bethe–Salpeter‡ scattering. Here the scattering amplitude had been shown (Wanders, 1960) to satisfy the Mandelstam representation only for each perturbative order in the ladder approximation. As far as the total Bethe–Salpeter scattering amplitude is concerned, neither fixed-t dispersion relation nor the Martin domain is known to exist. This is particularly distressing if we recall that single-dispersion relation and the Martin domain are, in general, derivable in axiomatic field theory. It is the purpose of this note to present simple reasonings showing that total Bethe–Salpeter scattering amplitude in the ladder approximation indeed satisfies fixed-t dispersion relation and has a Martin domain. It will be shown further that Mandelstam representation is valid in the same approximation.

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‡ For a general review see Nakanishi (1969).

For simplicity, we shall consider two nonidentical neutral scalar nucleons of equal mass m with incoming momenta p_1 and p_2, interacting through a neutral scalar meson of mass μ. We take the outgoing nucleon to have momenta p_3 and p_4 and define the invariant variables.

$$s = (p_1 + p_2)^2$$

$$t = (p_3 + p_1)^2$$

In the center-of-mass system of the two nucleons

$$s = 4(k^2 + m^2)$$

where k is the center-of-mass momentum of either nucleon. The center-of-mass motion of the two nucleons may be separated from their internal motion and the Bethe–Salpeter wave function $\psi(x,y)$ in ladder approximation can be written as

$$\psi(x,y) = \exp\left[i\left(\frac{p_1+p_2}{2}\right)(x+y)\right]\int d^4p\psi(p)\exp\left[ip(x-y)\right] \quad (1.1)$$

where $\psi(p)$ satisfies the Bethe–Salpeter integral equation,

$$\psi(p) = \delta\left(p - \frac{p_1-p_2}{2}\right) - \frac{ig^2}{(2\pi)^4}\left[\frac{1}{[(p_1+p_2)/2+p]^2+m^2-i\epsilon}\right.$$
$$\left.\frac{1}{[(p_1+p_2)/2-p]^2+m^2-i\epsilon}\int d^4p'\frac{\psi(p')}{(p-p')^2+\mu^2-i\epsilon}\right] \quad (1.2)$$

From the following Gaussian transform for the homogeneous term in the above integral equation,

$$\psi(p) = \delta\left(p - \frac{p_1-p_2}{2}\right) - \frac{ig^2}{(2\pi)^4}\int_0^\infty dx\int_0^\infty dy\int_0^\infty dzz^2 f(x,y,z,s)$$
$$\exp\left(-iz\left[\left\{\left(\frac{p_1+p_2}{2}+p\right)^2+m^2\right\}x+\left\{\left(\frac{p_1+p_2}{2}-p\right)^2+m^2\right\}y\right.\right.$$
$$\left.\left.+\left(p-\frac{p_1-p_2}{2}\right)^2+\mu^2\right]\right) \quad (1.3)$$

Okubo and Feldman (1960) had obtained an integral representation for the weight function $f(x,y,z,s)$,

$$f(x,y,z,s) = 1 + \frac{ig^2}{16\pi^2}\int_0^x du\int_0^x dv\int_0^\infty dw\frac{f(u,v,z+w,s)}{1+u+v}$$
$$\exp\left\{i\left[\left(\frac{suv-m^2(u+v)^2}{1+u+v}-\mu^2\right)w-\mu^2 z\left(1+\frac{z}{w}\right)(1+u+v)\right]\right\} \quad (1.4)$$

The Bethe–Salpeter scattering amplitude in the ladder approximation is given in terms of the weight function by

$$T(s,t) = -\frac{ig^2}{(2\pi)^3}\int_0^\infty dz f(z,s)\exp\left[-iz(t+\mu^2)\right] \tag{1.5}$$

where

$$f(z,s) = \lim_{\substack{x\to\infty\\ y\to\infty}} f(x,y,z,s) \tag{1.6}$$

We shall see in the following that the Okubo–Feldman representation, together with the unitarity condition contains all informations about the analyticity properties of the Bethe–Salpeter scattering amplitude in the ladder approximation.

2. *The Martin Domain*

The Okubo–Feldman integral representation for $f(x,y,z,s)$ had been considered by Wanders (1960) using iteration method. Using his notation, we write

$$f(x,y,z,s) = \sum_{n=1}^{\infty}\left(\frac{g^2}{16\pi^2}\right)^{n-1} f_n(x,y,z,s) \tag{2.1}$$

$$T(s,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{g^2}{16\pi^2}\right)^{n} T_n(s,t) \tag{2.2}$$

with

$$\begin{aligned} f_1 &= 1 \\ f_n(z,s) &= \lim_{\substack{x\to\infty\\ y\to\infty}} f_n(x,y,z,s) \end{aligned} \tag{2.3}$$

and

$$T_n(s,t) = -i\int_0^\infty dz f_n(z,s)\exp\left[-iz(t+\mu^2)\right] \tag{2.4}$$

Wanders had shown that for real $s < 4m^2$, the scattering amplitude $T_n(s,t)$ in the nth order iteration is regular in the whole t-plane except for a cut on the negative real axis from $t = -(n\mu)^2$ to $-\infty$ and is bounded by $c|t|^\alpha$ with $\alpha > 1/2$ and c, α independent of n. Since this is true for a finite segment of s, it will also be true for all s into which $T_n(s,t)$ can be analytically continued from $s < 4m^2$ real. This result clearly shows that the cut for $T_n(s,t)$ on the t-plane gets further away from the origin as the iteration order n gets larger. In particular, $T_n(s,t)$ is regular for $|t| < \mu^2$ for all iteration orders. Also, from Wanders' upper bound on $T_n(s,t)$, the iteration expansion converges uniformly for $g^2 < 16\pi^2$. It follows that for a given s, the total Bethe–Salpeter scattering amplitude in the ladder approximation is analytic in a non-shrinking Martin domain (Martin, 1966)

$$|t| < \mu^2 \tag{2.5}$$

Although this is a small analyticity domain compared to the whole t-plane cut from $-(n\mu)^2$ to $-\infty$ inside which $T_n(s,t)$ is analytic, it will be sufficient to derive the Mandelstam representation for $T(s,t)$ if the s-plane analyticity and unitarity condition are fully taken into account.

3. *Single-Dispersion Relation for $T(s,t)$ in the Cut s-Plane*

From the Okubo–Feldman representation equation (1.4), it is clear that $f(x,y,z,s)$ can be analytically continued into the upper half s-plane because in replacing s by $s_1 + is_2$ with $s_2 > 0$ and real, we get an extra exponential factor

$$\exp\left(\frac{s_2 uvw}{1+u+v}\right)$$

in the integrand of equation (1.4). From equations (1.5) and (1.6), it follows that $f(z,s)$ and $T(s,t)$ are likewise analytic inside the domain $\operatorname{Im} s > 0$.

In the lower half of s-plane, Wanders had also obtained an upper bound for $|T_n(s,t)|$ for each iteration order n

$$|T_n(s,t)| < \frac{1}{|\sin\theta|^n}\left[\frac{A_1}{|s-4m^2|^{n-3/2}} + \frac{A_2}{|s-4m^2|^{n-5/2}}\right]$$
$$\text{for } 0 < \theta < \pi/2 \text{ or } 3\pi/2 < \theta < 2\pi \tag{3.1}$$

and

$$|T_n(s,t)| < \left[\frac{A_1}{|s-4m^2|^{n-3/2}} + \frac{A_2}{|s-4m^2|^{n-5/2}}\right] \qquad \text{for } \pi/2 \leqslant \theta \leqslant 3\pi/2 \tag{3.2}$$

where $\theta = \arg(s-4m^2)$ and A_1 and A_2 are energy-independent constants. One can consider the sum of the iterative series and find the following upper bounds,

$$|T(s,t)| < [|s-4m^2|^{3/2} A_1 + |s-4m^2|^{5/2} A_2] \sum_{n=0}^{\infty} \left(\frac{g^2}{16\pi^2}\right)^2 \frac{1}{|s-4m^2|^n}$$
$$\text{for } \pi < \theta < 3\pi/2 \tag{3.3}$$

and

$$|T(s,t)| < [|s-4m^2|^{3/2} A_1 + |s-4m^2|^{5/2} A_2] \sum_{n=1}^{\infty} \left(\frac{g^2}{16\pi^2}\right)^n \frac{1}{|\sin\theta|^n\,|s-4m^2|^n}$$
$$\text{for } 3\pi/2 < \theta < 2\pi \tag{3.4}$$

For illustration, let us take $g = 4\pi$. It is then clear from the above bounds that $T(s,t)$ is analytic in the lower half of s-plane except for a unit semicircle around $s = 4m^2$ and a region extended from $s = -im^2$ to $\infty - i\epsilon$ as defined by

$$|\sin\theta|\,|s-4m^2| \leqslant 1, \qquad 3\pi/2 < \theta < 2\pi \tag{3.5}$$

It will be difficult to get a larger analyticity domain for $T(s,t)$ from the iterative series. To proceed further, we go back to the Okubo–Feldman representation in equation (1.4). We notice that $f(x,y,z,s)$ can be analytically continued into the lower half z-plane since in replacing z by $z_1 - iz_2$ with $z_2 > 0$ and real, we get the following exponentially falling factors

$$\exp\left[-\mu^2 z_2(1+u+v) - \frac{2\mu^2}{w}(z_1 z_2)(1+u+v)\right]$$

in the integrand of equation (1.4). $z = 0$ is a branch point for $f(x,y,z,s)$, but we can make a cut on the negative real axis $-\infty < z < 0$, $f(x,y,z_1 - iz_2,s)$ is then regular in the lower half z-plane but not on the cut. This z-plane analyticity implies that the weight function $f(x,y,z_1 - iz_2,s)$ in the lower half z-plane is completely determined, for instance, by its values along the negative imaginary axis. Now taking z purely imaginary and negative and $s < 2m^2$, the w-integration in the Okubo–Feldman representation may be rotated to the negative real axis through the lower half of w-plane because in replacing w by $w_1 - iw_2$ with $w_2 > 0$ and real, we again get exponentially falling factors

$$\exp\left[-\frac{\mu^2 z_2^2 w_2}{w_1 + w_2^2}(1+u+v) - w_2\frac{(2m^2-s)uv + m^2(u^2+v^2) + \mu^2(1+u+v)}{1+u+v}\right]$$

in the integrand of equation (1.4). In performing the above rotation $z = -iz_2 - w$ lies in the lower half of z-plane and hence $f(x,y,-iz_2 - w,s)$ is always inside its analyticity domain. At the lower integration limit near $w = 0$, but with w lying in the lower half-plane, the behavior of the integrand is also regular because for z-imaginary, the factor

$$\exp\left[-i\frac{\mu^2 z^2}{w}(1+u+v)\right]$$

will go to zero exponentially as $w \to 0$. After such a rotation of the w-integration, the Okubo–Feldman representation becomes

$$f(x,y,z,s) = 1 + \frac{ig^2}{16\pi^2}\int_0^x du \int_0^y dv \int_0^\infty dw \frac{f(u,v,z-w,s)}{1+u+v}$$
$$\exp\left\{i\left[\frac{suv - m^2(u+v)^2}{1+u+v} - \mu^2\right](-w) - \mu^2 z\left(1-\frac{z}{w}\right)(1+u+v)\right\} \tag{3.6}$$

Since this integral equation is valid in a finite domain in s and z, it holds everywhere within the analytic regions of $f(x,y,z,s)$. If s is now replaced by $s_1 - is_2$, we will get an exponential damping factor

$$\exp\left(\frac{s_2 uv}{1+u+v}w\right)$$

in the integrand of equation (3.6). This means that $f(x,y,z,s)$ can be analytically continued into the lower-half s-plane. This, together with previous results means that $f(x,y,z,s)$ and hence $f(z,s)$ and $T(s,t)$ are analytic in the entire s-plane with a cut from $4m^2$ to ∞. It follows that $T(s,t)$ satisfies the following dispersion relation,

$$T(s,t) = \sum_{n=0}^{N-1} a_n s^n + \frac{S^N}{\pi} \int_{4m^2}^{\infty} ds' \frac{\operatorname{Im} T(s',t)}{s'^N(s'-s-i\epsilon)} \tag{3.7}$$

4. *Mandelstam Representation for* $T(s,t)$

We shall now consider the effect of the elastic unitarity condition, which is applicable for all real $s \geqslant 4m^2$ for the Bethe–Salpeter amplitude in the ladder approximation and can be written as

$$\operatorname{Im} T(s,t) = \frac{\theta(s-4m^2)}{4\pi}\sqrt{(s-4m^2)}\int T(s,\cos\theta')\,T^*(s,\cos\theta'')\,d\Omega' \tag{4.1}$$

where

$$\cos\theta'' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi'-\phi)$$

Whenever applicable, the elastic unitarity has the following consequence: Suppose that for a given real $s \geqslant 4m^2$, $T(s,t)$ is analytic in the t-plane inside an ellipse $E(0,-4k^2|r)$ with foci at $t=0$ and $-4k^2$ and right-hand extremity at $t=r$ then $\operatorname{Im} T(s,t)$ will be analytic inside a confocal but larger ellipse

$$E\left(0,-4k^2|4r\left(1+\frac{r}{4k^2}\right)\right) \tag{4.2}$$

for the same value of k^2. This important property of the elastic unitarity had been utilized previously in proving the validity of Mandelstam representation in potential scattering (Bowcock & Martin, 1959; Blankenbecler *et al.*, 1960; Cheung, 1969) and also in proving the necessity of N-particle production amplitudes in field theory (Cheung & Toll, 1967; Cheung, 1968). In all these cases, a scattering amplitude was shown to satisfy a Mandelstam representation if elastic unitarity is applicable for all real $s \geqslant 4m^2$ and if the following two conditions are fulfilled.

(a) The scattering amplitude has a nonshrinking Martin analyticity domain in the momentum-transfer t-plane.

(b) For a given t inside the Martin domain, the scattering amplitude has a single-dispersion relation in the cut s-plane.

We can now apply this lemma to the problem under consideration. We shall refer interested readers to Bowcock & Martin (1959), Blankenbecler *et al.* (1960) and Cheung (1969) for details. Since we have shown in the above that all the necessary conditions are satisfied, we reach the conclusion that the total Bethe–Salpeter scattering amplitude in the ladder approximation has the following Mandelstam representation,

$$T(s,t) = \sum_{n=0}^{N-1} a_n t^n + \sum_{m=0}^{M-1} b^m s^m + \frac{t^N s^M}{\pi^2} \int_{\mu^2}^{\infty} dt' \int_{4m^2}^{\infty} ds' \frac{\rho(s',t')}{s'^M t'^N (s'-s)(t'-t)}. \tag{4.3}$$

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